Quiz #1'

Date: September 19, 2025

Question 1. 5 points

Identify the error in the following proof, and explain why it is incorrect.

Proposition: For all $n \geqslant 2$, the number n is even.
Proof: We proceed by strong induction.

Base case: The base case $n = 2$ is true, as 2 is even.

Strong induction hypothesis: Assume that every number less than n is even. Then, as $n = (n-2) + 2$, and $n - 2$ is is less than n, so even by the strong induction hypothesis, we deduce that n is even, as desired. ■

Solution: The error is that in the proof of the strong induction hypothesis, we implicitly used that $n - 2 \geqslant 2$ (so that our statement applies). In particular, we need to actually check it for the base cases $n = 2$ and $n = 3$, and it fails for $n = 3$.

Rubric:

- **3 points:** identifying that it's an issue with base cases,
- **2 points:** explicating the precise issue.

Question 2. 10 points

1. **(8 points)** Let $r \neq 1$ be a real number. Prove that for all $n \geqslant 1$ the equality
$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}, \tag{1}$$
holds.
2. **(2 points)** Formulate a version of Equation (1) which works even when $r = 1$, and justify this one missing case.

Solution:

1. We proceed by induction.

Base case: The base case is when $n = 1$ in which case the equality becomes

$$1 + r = \frac{r^2 - 1}{r - 1},$$

which is true as $r^2 - 1 = (r - 1)(r + 1)$.

Induction hypothesis: Assume that the equality

$$1 + r + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},$$

holds. Then, adding r^{n+1} to both sides yields

$$\begin{aligned} 1 + \cdots + r^n + r^{n+1} &= \frac{r^{n+1} - 1}{r - 1} + r^{n+1} \\ &= \frac{r^{n+1} - 1 + r^{n+1}(r - 1)}{r - 1} \\ &= \frac{r^{n+1} - 1 + r^{n+2} - r^{n+1}}{r - 1} \\ &= \frac{r^{n+2} - 1}{r - 1}, \end{aligned}$$

as desired.

2. One may instead write

$$(r - 1)(1 + \cdots + r^n) = r^{n+1} - 1.$$

For $r \neq 1$ this reduces to (1), and when $r = 1$ both the left-hand and right-hand sides are 0.

Rubric:

1.
   - **2 points:** correctly setting up the proof, i.e., writing "We proceed by induction", and then having a base case and induction hypothesis step.
   - **2 point:** correctly explaining the base case,
   - **4 points:** correctly explaining the inductive hypothesis step.
2.
   - **1 points:** correctly recognizing the correct equation
   - **1 points:** observing that both sides are 0 when $r = 1$.